Continuous variable entanglement swapping

2004 J. Phys. A: Math. Gen. 37 L385

(http://iopscience.iop.org/0305-4470/37/31/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) L385–L390 PII: S0305-4470(04)81843-2

LETTER TO THE EDITOR

Continuous variable entanglement swapping

Masashi Ban

Advanced Research Laboratory, Hitachi Ltd, 2520 Akanuma, Hatoyama, Saitama 350-0395, Japan
and
CREST, Japan Science and Technology Agency, 1-1-9 Yaesu, Chuo-ku, Tokyo 103-0028, Japan

Received 8 June 2004
Published 21 July 2004
Online at stacks.iop.org/JPhysA/37/L385
doi:10.1088/0305-4470/37/31/L01

Abstract
A sequence of entanglement swapping of continuous variables is considered. It is classified into one-way entanglement swapping and two-way entanglement swapping, where the former (the latter) uses one-way (two-way) classical communication. When resources of quantum entanglement are bipartite Gaussian states, it is shown that the one-way entanglement swapping is superior to the two-way entanglement swapping. This means that although the entanglement swapping is performed the same number of times, there is a case that the one-way entanglement swapping can yield an entangled state while the two-way entanglement swapping cannot.

PACS numbers: 03.67.Hk, 03.67.Mn, 03.67.−a

Nonlocal entanglement between quantum states is a useful resource that has a number of applications in quantum information processing [1, 2], such as a secret key distribution [3], quantum teleportation [4, 5] and superdense coding [6–10]. Entanglement swapping enables two parties that do not share quantum entanglement to share quantum entanglement with the assistance of a third party [11–16]. Furthermore, entanglement swapping may yield long-distance quantum entanglement from properly distributed short-distance quantum entanglement. Hence entanglement swapping is one of the important methods in quantum communication technology which aims at quantum information network. Entanglement swapping has been investigated for discrete [11–13] and continuous variables [14–16]. It may be considered as the quantum teleportation of a part of an entangled state [17, 18]. The purpose of this letter is to investigate the property of a sequence of continuous-variable entanglement swapping. It will be shown that when resources of quantum entanglement are bipartite Gaussian states, one-way entanglement swapping is better than two-way entanglement swapping for distant users to share quantum entanglement. As an example, the condition for a sequence of entanglement swapping to yield an entangled state is demonstrated by means of two-mode squeezed-vacuum states.

The entanglement swapping is equivalent to the quantum teleportation for a part of a bipartite quantum state. Hence we first summarize the general theory of the continuous-variable quantum teleportation with the standard protocol [19] in the convenient way for investigating entanglement swapping. To teleport from Alice to Bob an unknown quantum state $\hat{\rho}_{\rm in}$ which is defined on an infinite-dimensional Hilbert space $\mathcal{H}$, we assume that they share an arbitrary bipartite quantum state $\hat{W}^{\rm AB}$ defined on a tensor product of the infinite-dimensional Hilbert spaces $\mathcal{H}^{\rm A} \otimes \mathcal{H}^{\rm B}$. In the standard protocol, Alice performs the simultaneous measurement of the position and momentum for the compound system consisting of the input system in the quantum state $\hat{\rho}_{\rm in}$ and her part of the bipartite system in the quantum state $\hat{W}^{\rm AB}$. After the measurement, Alice informs Bob of the measurement outcome (x, p) by means of a classical communication channel. Knowing the measurement outcome, Bob applies the unitary transformation $\hat{D}(x, p)$ to his part of the bipartite system, where $\hat{D}(x, p)$ is the displacement operator

$$\hat{D}(x, p) = \exp[{\rm i}(p\hat{x} - x\hat{p})] = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \hat{D}(\alpha) \tag{1}$$

with $(\hat{a}, \hat{a}^\dagger)$ being bosonic annihilation and creation operators and $(\hat{x}, \hat{p})$ being canonical position and momentum operator which are related by the equation $\hat{a} = (\hat{x} + {\rm i}\hat{p})/\sqrt{2}$. The real and complex parameters are also related by $\alpha = (x + {\rm i}p)/\sqrt{2}$.

The continuous-variable quantum teleportation enables Bob to finally get the quantum state $\hat{\rho}_{\rm out}$ [19],

$$\hat{\rho}_{\rm out} = \int_{-\infty}^{\infty} {\rm d}x \int_{-\infty}^{\infty} {\rm d}p\, P(x, p)\hat{D}(x, p)\hat{\rho}_{\rm in}\hat{D}^\dagger(x, p) \tag{2}$$

where the function $P(x, p)$ is given by

$$P(x, p) = \langle\Psi|[\hat{1} \otimes \hat{D}(x, p)]^\dagger \hat{W}^{\rm AB}[\hat{1} \otimes \hat{D}(x, p)]|\Psi\rangle \tag{3}$$

and where $|\Phi\rangle$ is the continuous version of the EPR–Bell state

$$|\Phi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} {\rm d}x|x\rangle \otimes |x\rangle \tag{4}$$

with $|x\rangle$ being an eigenstate of the canonical position operator $(\hat{x}|x\rangle = x|x\rangle)$. The bipartite quantum state $\hat{W}^{\rm AB}$ shared by Alice and Bob can be represented in terms of the two-mode Wigner function $W(z_1, z_2)$ of the complex variables z_1 and z_2 [20, 21],

$$\hat{W}^{\rm AB} = 4\int \frac{{\rm d}^2z_1}{\pi} \int \frac{{\rm d}^2z_2}{\pi} W(z_1, z_2)(-1)^{(\hat{a}_1^\dagger - z_1^*)(\hat{a}_1 - z_1)} \otimes (-1)^{(\hat{a}_2^\dagger - z_2^*)(\hat{a}_2 - z_2)} \tag{5}$$

where $\left(\hat{a}_1, \hat{a}_1^\dagger\right)$ and $\left(\hat{a}_2, \hat{a}_2^\dagger\right)$ are bosonic annihilation and creation operators of the two modes. Then, substituting this equation into equation (3), we find that the function $P(x, p)$ is expressed in terms of the Wigner function $W(z_1, z_2)$ as

$$P(x, p) = \frac{1}{2\pi} \int \frac{{\rm d}^2z}{\pi} W(z^* - \alpha^*, z) \equiv \frac{1}{2\pi}\mathcal{W}(\alpha) \qquad \left(\alpha = \frac{x + {\rm i}p}{\sqrt{2}}\right). \tag{6}$$

We suppose that the bipartite quantum state $\hat{W}^{\rm AB}$ shared by Alice and Bob is a two-mode Gaussian state, the characteristic function $C(z_1, z_2)$ of which is given by

$$C(z_1, z_2) = {\rm Tr}[\hat{D}(z_1) \otimes \hat{D}(z_2)\hat{W}^{\rm AB}] = \exp\left(-\tfrac{1}{2}\mathsf{z}^\dagger\mathsf{V}\mathsf{z}\right) \tag{7}$$

where $\mathsf{z} = (z_1^*, z_1, z_2^*, z_2)^\dagger$ and V is a 4×4 Hermitian matrix. The corresponding Wigner function $W(z_1, z_2)$ is given by

$$W(z_1, z_2) = \sqrt{\det\mathsf{W}}\exp\left(-\tfrac{1}{2}\mathsf{z}^\dagger\mathsf{W}\mathsf{z}\right) \tag{8}$$

where the 4×4 Hermitian matrix W is related to V by the relation $\mathsf{W} = \mathsf{E}\mathsf{V}^{-1}\mathsf{E}$ with the diagonal matrix $\operatorname{diag}\mathsf{E} = (1, -1, 1, -1)$. In the following, we assume that the matrix V is a simple but important form [22]

$$\mathsf{V} = \begin{pmatrix} \bar{n}+\frac{1}{2} & 0 & 0 & \bar{m} \\ 0 & \bar{n}+\frac{1}{2} & \bar{m}^* & 0 \\ 0 & \bar{m} & \bar{n}+\frac{1}{2} & 0 \\ \bar{m}^* & 0 & 0 & \bar{n}+\frac{1}{2} \end{pmatrix} \tag{9}$$

where $\bar{n} = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle_{12} = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle_{12}$ and $\bar{m} = -\langle \hat{a}_1 \hat{a}_2 \rangle_{12}$ with $\langle \cdots \rangle_{12} = \mathrm{Tr}[\cdots \hat{W}_{12}]$. The Wigner function $W(z_1, z_2)$ represents the mixed EPR–Bell state. For instance, when Alice and Bob share a two-mode squeezed-vacuum state through a noisy quantum channel, the parameters $\bar{n}$ and $\bar{m}$ are given by $\bar{n} = (1/2)T(\cosh 2r - 1) + (1 - T)\bar{n}_{\mathrm{th}}$ and $\bar{m} = -(1/2)T \sinh 2r$, where r is the squeezing parameter, T is the transmission rate and $\bar{n}_{\mathrm{th}}$ is the average photon number of the channel noise. The Heisenberg uncertainty relation requires that the parameters $\bar{n}$ and $\bar{m}$ satisfy the inequality $\bar{n}(\bar{n}+1) \geqslant |\bar{m}|^2$. Furthermore the two-mode Gaussian state $\hat{W}$ is separable if and only if the parameters $\bar{n}$ and $\bar{m}$ satisfy the inequality $\bar{n} \geqslant |\bar{m}|$ [23–25].

When the matrix V is given by equation (9), the two-mode Wigner function $W(z_1, z_2)$ is obtained from equation (8)

$$W(z_1, z_1) = \frac{1}{\left(\bar{n}+\frac{1}{2}\right)^2 - |\bar{m}|^2} \exp\left[-\frac{\Phi(z_1, z_2)}{\left(\bar{n}+\frac{1}{2}\right)^2 - |\bar{m}|^2}\right] \tag{10}$$

where the function $\Phi(z_1, z_2)$ is given by

$$\Phi(x_1, z_2) = \left(\bar{n}+\tfrac{1}{2}\right)(|z_1|^2 + |z_2|^2) + \bar{m}^* z_1 z_2 + \bar{m} z_1^* z_2^*. \tag{11}$$

Thus, substituting equation (10) into equation (6), we obtain the function $\mathcal{W}(\alpha)$ which determines the continuous-variable quantum teleportation

$$\mathcal{W}(\alpha) = \frac{1}{\delta_{\bar{n}\bar{m}}} \exp\left(-\frac{|\alpha|^2}{\delta_{\bar{n}\bar{m}}}\right) \tag{12}$$

with $\delta_{\bar{n}\bar{m}} = 2\bar{n} + \bar{m} + \bar{m}^* + 1$. The inequality $\delta_{\bar{n}\bar{m}} \geqslant 0$ holds due to the uncertainty relation. When the two-mode Gaussian state $\hat{W}^{\mathrm{AB}}$ is separable, the inequality $\delta_{\bar{n}\bar{m}} \geqslant 1$ is satisfied.

We now consider a sequence of a continuous-variable entanglement swapping, the resources of which are arbitrary but identical bipartite quantum states. Note that the entanglement swapping is equivalent to the quantum teleportation of a part of the bipartite quantum state. In fact, we suppose that Alice and Charlie share an entangled state $\hat{W}^{\mathrm{AC}}$ and Charlie and Bob share another entangled state $\hat{W}^{\mathrm{BC}}$. When Charlie teleports his part of the entangled state $\hat{W}^{\mathrm{AC}}$ to Bob by means of the entangled state $\hat{W}^{\mathrm{BC}}$, Alice and Bob can share the bipartite quantum state as a result. This is nothing but the entanglement swapping. Hence we can apply the results summarized above to continuous-variable entanglement swapping. A sequence of entanglement swapping may be classified into one-way entanglement swapping and two-way entanglement swapping which are depicted in figure 1. Classical information of the measurement outcomes flows from the left to the right in the one-way entanglement swapping (see figure 1(*a*)) while it flows in both directions in the two-way entanglement swapping (see figure 1(*b*)). We suppose that the entanglement swapping is performed by means of $2k+1$ identical bipartite quantum states. Here we assume that the two-way entanglement swapping begins at the bipartite quantum state located at the mid-point, so that Alice and Bob can share the symmetric bipartite state. It is obvious that the one-way entanglement swapping yields the asymmetric bipartite state. Let $\hat{W}^{\mathrm{AB}}_{\text{one-way}}$ and $\hat{W}^{\mathrm{AB}}_{\text{two-way}}$ be the bipartite quantum states

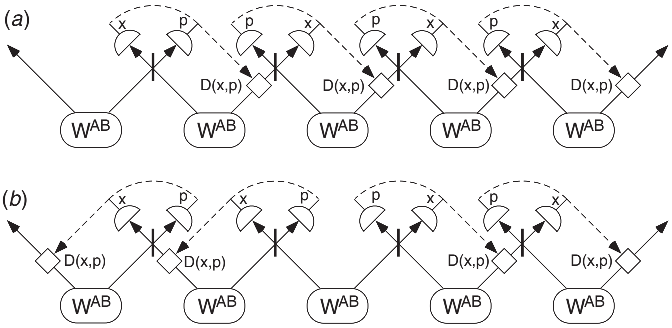

Figure 1. The schematic representation of the sequence of continuous-variable entanglement swapping, where (*a*) shows the one-way entanglement swapping and (*b*) shows the two-way entanglement swapping.

that Alice and Bob can share in average by the one-way and two-way entanglement swapping. These quantum states are given by

$$\hat{W}^{\mathrm{AB}}_{\text{one-way}} = (\hat{\mathcal{I}} \otimes \hat{\mathcal{L}}^{2k})\hat{W}^{\mathrm{AB}} \qquad \hat{W}^{\mathrm{AB}}_{\text{two-way}} = (\hat{\mathcal{L}}^{k} \otimes \hat{\mathcal{L}}^{k})\hat{W}^{\mathrm{AB}} \tag{13}$$

where the trace-preserving completely positive map is defined by

$$\hat{\mathcal{L}}\hat{X} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p\, P(x, p)\hat{D}(x, p)\hat{X}\hat{D}^{\dagger}(x, p). \tag{14}$$

In the following, we assume that the bipartite quantum state $\hat{W}^{\mathrm{AB}}$ is the Gaussian state which is characterized by equation (9) and hence we have equation (12).

To investigate the property of the bipartite Gaussian states $\hat{W}^{\mathrm{AB}}_{\text{one-way}}$ and $\hat{W}^{\mathrm{AB}}_{\text{two-way}}$ shared by Alice and Bob after the entanglement swapping, we calculate the characteristic functions (7) of these states and then we obtain from equations (12)–(14),

$$C_{\text{one-way}}(z_1, z_2) = \exp\left[-\tfrac{1}{2}\mathsf{z}^{\dagger}\mathsf{V}(\bar{n}, \bar{n} + 2k\delta_{\bar{n}\bar{m}})\mathsf{z}\right] \tag{15}$$

$$C_{\text{two-way}}(z_1, z_2) = \exp\left[-\tfrac{1}{2}\mathsf{z}^{\dagger}\mathsf{V}(\bar{n} + k\delta_{\bar{n}\bar{m}}, \bar{n} + k\delta_{\bar{n}\bar{m}})\mathsf{z}\right] \tag{16}$$

where the 4×4 Hermitian matrix $\mathsf{V}(\bar{n}_1, \bar{n}_2)$ is defined by

$$\mathsf{V}(\bar{n}_1, \bar{n}_2) = \begin{pmatrix} \bar{n}_1 + \frac{1}{2} & 0 & 0 & \bar{m} \\ 0 & \bar{n}_1 + \frac{1}{2} & \bar{m}^* & 0 \\ 0 & \bar{m} & \bar{n}_2 + \frac{1}{2} & 0 \\ \bar{m}^* & 0 & 0 & \bar{n}_2 + \frac{1}{2} \end{pmatrix}. \tag{17}$$

The necessary and sufficient condition for the separability of the bipartite Gaussian state characterized by the matrix $\mathsf{V}(\bar{n}_1, \bar{n}_2)$ is given by [23–25]

$$\bar{n}_1\bar{n}_2 \geqslant |\bar{m}|^2. \tag{18}$$

Hence the bipartite Gaussian state $\hat{W}^{\mathrm{AB}}_{\text{one-way}}$ obtained by the one-way entanglement swapping is separable if and only if

$$\bar{n}(\bar{n} + 2k\delta_{\bar{n}\bar{m}}) \geqslant |\bar{m}|^2 \tag{19}$$

while the bipartite Gaussian state $\hat{W}^{\mathrm{AB}}_{\text{two-way}}$ obtained by the two-way entanglement swapping becomes separable if and only if

$$(\bar{n} + k\delta_{\bar{n}\bar{m}})^2 \geqslant |\bar{m}|^2. \tag{20}$$

It is obvious that the inequality $\bar{n}(\bar{n} + 2k\delta_{\bar{n}\bar{m}}) < (\bar{n} + k\delta_{\bar{n}\bar{m}})^2$ holds. This means that the parameters $\bar{n}$ and $\bar{m}$ can satisfy the inequality

$$\bar{n}(\bar{n} + 2k\delta_{\bar{n}\bar{m}}) < |\bar{m}|^2 \leqslant (\bar{n} + k\delta_{\bar{n}\bar{m}})^2. \tag{21}$$

When the parameters of the bipartite Gaussian state satisfy this inequality, Alice and Bob can share the entangled Gaussian state by the one-way entanglement swapping though they cannot if the two-way entanglement swapping is applied. It is important to note that the entanglement swapping is performed the same number of times in the both one-way and two-way entanglement swapping. Thus the one-way entanglement swapping is superior to the two-way entanglement swapping for sharing the entangled state. The entangled state created by the one-way entanglement swapping is asymmetric. The continuous-variable quantum teleportation by means of the asymmetric entangled state has been investigated [26].

The Heisenberg uncertainty relation requires that the parameters $\bar{n}$ and $\bar{m}$ satisfy the inequality $\bar{n}(\bar{n} + 1) \geqslant |\bar{m}|$. In the extreme case, we have $\bar{m} = \bar{m}^* = -\sqrt{\bar{n}(\bar{n}+1)}$ and thus $\delta_{\bar{n}\bar{m}} = 2\bar{n} + \bar{m} + \bar{m}^* + 1 = (\sqrt{\bar{n}+1} - \sqrt{\bar{n}})^2$. In this case, the inequality equation (19) becomes

$$\bar{n} \leqslant \frac{(2k-1)^2}{8k} \equiv \bar{n}_1 \tag{22}$$

and the inequality equation (20) becomes

$$\bar{n} \leqslant \frac{k^2}{2k+1} \equiv \bar{n}_2. \tag{23}$$

Then we can obtain the following result:

	$\bar{n} \leqslant \bar{n}_1$	$\bar{n}_1 < \bar{n} \leqslant \bar{n}_2$	$\bar{n}_2 < \bar{n}$
$\hat{W}^{\mathrm{AB}}_{\text{one-way}}$	Separable	Entangled	Entangled
$\hat{W}^{\mathrm{AB}}_{\text{two-way}}$	Separable	Separable	Entangled

We suppose that the bipartite Gaussian state $\hat{W}^{\mathrm{AB}}$ is a two-mode squeezed-vacuum state with real squeezing parameter r. In this case, we have $\bar{n} = (1/2)(\cosh 2r - 1)$ and $\bar{m} = -(1/2)\sinh 2r$ which satisfy the equality $\bar{m} = \bar{m}^* = -\sqrt{\bar{n}(\bar{n}+1)}$. Hence we obtain figure 2 and the following result:

	$r \leqslant \frac{1}{2}\ln(2k)$	$\frac{1}{2}\ln(2k) < \bar{n} \leqslant \frac{1}{2}\ln(2k+1)$	$\frac{1}{2}\ln(2k+1) < \bar{n}$
$\hat{W}^{\mathrm{AB}}_{\text{one-way}}$	Separable	Entangled	Entangled
$\hat{W}^{\mathrm{AB}}_{\text{two-way}}$	Separable	Separable	Entangled

When we perform the entanglement swapping twice ($k = 1$), the one-way entanglement swapping needs $r = 0.347$ of squeezing to yield the entangled state while the two-way entanglement requires $r = 0.549$ of squeezing.

In summary, we have considered the sequence of the continuous-variable entanglement swapping that is classified into the one-way entanglement swapping and the two-way entanglement swapping. The one-way entanglement swapping yields the asymmetric bipartite state and the two-way entanglement swapping gives the symmetric bipartite state. It has been shown that there is the parameter region in which the one-way entanglement swapping provides the entangled state while the two-way entanglement swapping cannot, though the entanglement swapping is performed the same number of times in the both one-way and

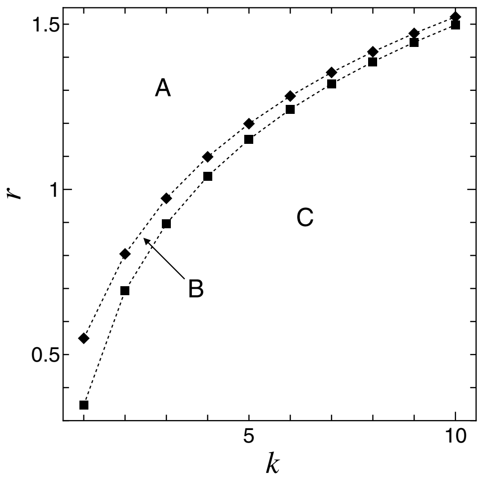

Figure 2. The relation between the strength of the squeezing parameter r that is necessary for the inseparability and the parameter k, where $2k$ times the entanglement swapping is performed. Both the bipartite states $\hat{W}^{AB}_{\text{one-way}}$ and $\hat{W}^{AB}_{\text{two-way}}$ become inseparable in the region (A) and separable in the region (C). The bipartite state $\hat{W}^{AB}_{\text{one-way}}$ is inseparable and $\hat{W}^{AB}_{\text{two-way}}$ is separable in the region (B).

two-way entanglement swapping. This means that the one-way entanglement swapping is better than the two-way entanglement swapping for sharing the entangled state.

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Braunstein S L and Pati A K (eds) 2003 *Quantum Information with Continuous Variables* (Dordrecht: Kluwer)
[3] Ekert A K 1991 *Phys. Rev. Lett.* **67** 661
[4] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[5] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[6] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
[7] Ban M 1999 *J. Opt.* B **1** L9
[8] Braunstein S L and Kimble H J 2000 *Phys. Rev.* A **61** 042302
[9] Ban M 2000 *J. Opt.* B **2** 786
[10] Ban M 2000 *Phys. Lett.* A **276** 213
[11] Zukowski M, Zeilinger A, Horne M A and Ekert A K 1993 *Phys. Rev. Lett.* **71** 4287
[12] Bose S, Vedral V and Knight P L 1998 *Phys. Rev.* A **57** 822
[13] Bose S, Vedral V and Knight P L 1999 *Phys. Rev.* A **60** 194
[14] Polkinghorne R E S and Ralph T C 1999 *Phys. Rev. Lett.* **83** 2095
[15] Tan S M 1999 *Phys. Rev.* A **60** 2752
[16] Lock P and Braunstein S L 1999 *Phys. Rev.* A **61** 010302(R)
[17] Bennett C H, Brassard G, Crepeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[18] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[19] Ban M, Takeoka M and Sasaki M 2002 *J. Phys. A: Math. Gen.* **35** L401
[20] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
[21] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
[22] Englert B and Wókiewicz K 2003 *Phys. Rev.* A **65** 054303
[23] Duan L, Giedke G, Cirac J I and Zoller P 2000 *Phys. Rev. Lett.* **84** 2722
[24] Simon R 2000 *Phys. Rev. Lett.* **84** 2726
[25] Englert B and Wódkiewicz K 2003 *Int. J. Quantum Inf.* **1** 153
[26] Kim M S and Lee J 2001 *Phys. Rev.* A **64** 012309